Chapter 14 Time-Varying Volatility and ARCH Models

Chapter Contents

Time-Varying Volatility and ARCH Models

- The nonstationary nature of the variables studied earlier implied that they had means that change over time
- Now we are concerned with stationary series, but with conditional variances that change over time
  - The model is called the autoregressive conditional heteroskedastic (ARCH) model
  - Financial time series have characteristics that are well represented by models with dynamic variances

14.1 The ARCH Model 1 of 7

- Consider a model with an AR(1) error term:
  - (14.1a) $y_t = \phi + e_t$
  - (14.1b) $e_t = \rho e_{t-1} + v_t, \qquad |\rho| < 1$
  - (14.1c) $v_t \sim N(0,\ \sigma_v^2)$

14.1 The ARCH Model 2 of 7

- The **unconditional mean** of the error is:

$$E[e_t] = E[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots] = 0$$

- The **conditional mean** for the error is:

$$E[e_t | I_{t-1}] = E[\rho e_{t-1} | I_{t-1}] + E[v_t] = \rho e_{t-1}$$

14.1 The ARCH Model 3 of 7

- The **unconditional variance** of the error is:

$$E\left[e_t - 0\right]^2 = E\left[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots\right]^2$$
$$= E\left[v_t^2 + \rho^2 v_{t-1}^2 + \rho^4 v_{t-2}^2 + \cdots\right]$$
$$= \sigma_v^2\left[1 + \rho^2 + \rho^4 + \cdots\right]$$
$$= \frac{\sigma_v^2}{1-\rho^2}$$

- The conditional variance for the error is:

$$E[(e_t - \rho e_{t-1})^2 | I_{t-1}] = E[v_t^2 | I_{t-1}] = \sigma_v^2$$

14.1 The ARCH Model 4 of 7

- Suppose that instead of a conditional mean that changes over time we have a conditional variance that changes over time
  - Consider a variant of the above model:
  - (14.2a) $y_t = \beta_0 + e_t$
  - (14.2b) $e_t | I_{t-1} \sim N(0, h_t)$
  - (14.2c) $h_t = \alpha_0 + \alpha_1 e_{t-1}^2, \qquad \alpha_0 > 0, \qquad 0 \leq \alpha_1 < 1$

14.1 The ARCH Model 5 of 7

- Equations (14.2b and 14.2c) describe the ARCH class of models
- The second equation (14.2b) says that the error term is conditionally normal
  - where I_{t-1} represents the information available at time $t - 1$ with mean 0 and time-varying variance, denoted as h_t
- The third equation (14.2c) models $\boldsymbol{h_t}$ as a function of a constant term and the lagged error squared

14.1 The ARCH Model 6 of 7

- The name — ARCH — conveys the fact that we are working with time-varying variances (heteroskedasticity) that depend on (are conditional on) lagged effects (autocorrelation)
  - This particular example is an ARCH(1) model

14.1 The ARCH Model 7 of 7

- The standardized errors are standard normal:
  - $\left(\frac{e_t}{\sqrt{h_t}} | I_{t-1}\right) = z \sim N(0,1)$
- We can write:
  - $E(e_t) = E(z_t)E\left(\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}\right)$
- And
  - $E(e_t^2) = E(z_t^2)E(\alpha_0 + \alpha_1 e_{t-1}^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2)$

14.2 Time-Varying Volatility 1 of 2

- The ARCH model has become a popular one because its variance specification can capture commonly observed features of the time series of financial variables
  - It is useful for modeling **volatility** and especially changes in volatility over time

EXAMPLE 14.1 Characteristics of Financial Variables

- The values of these series change rapidly from period to period in an apparently unpredictable manner; we say the series are volatile
- There are periods when large changes are followed by further large changes and periods when small changes are followed by further small changes
- Distributions where there are more observations around the mean and in the tails are said to be **leptokurtic**

FIGURE 14.1 Time series of returns to stock indices.

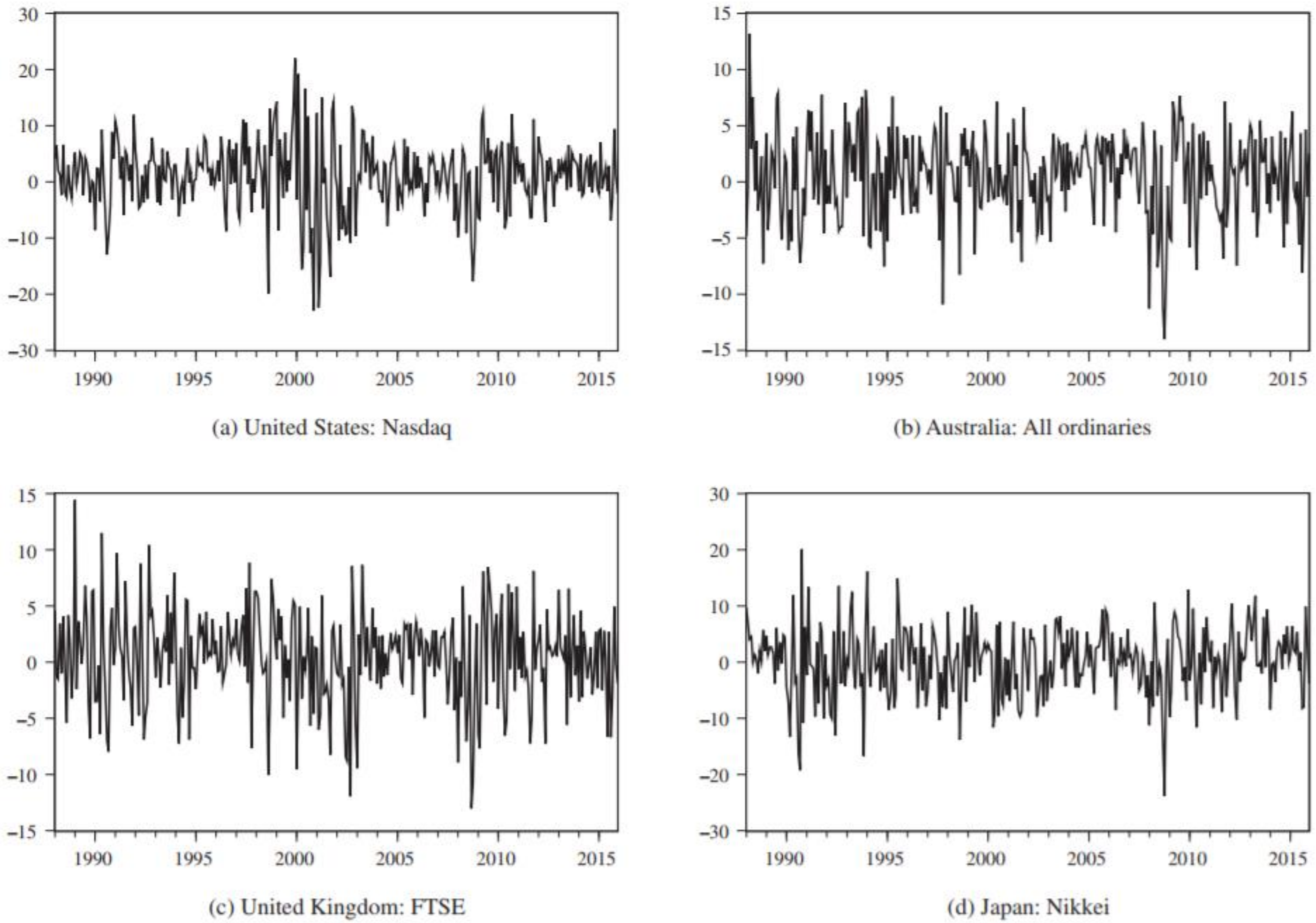
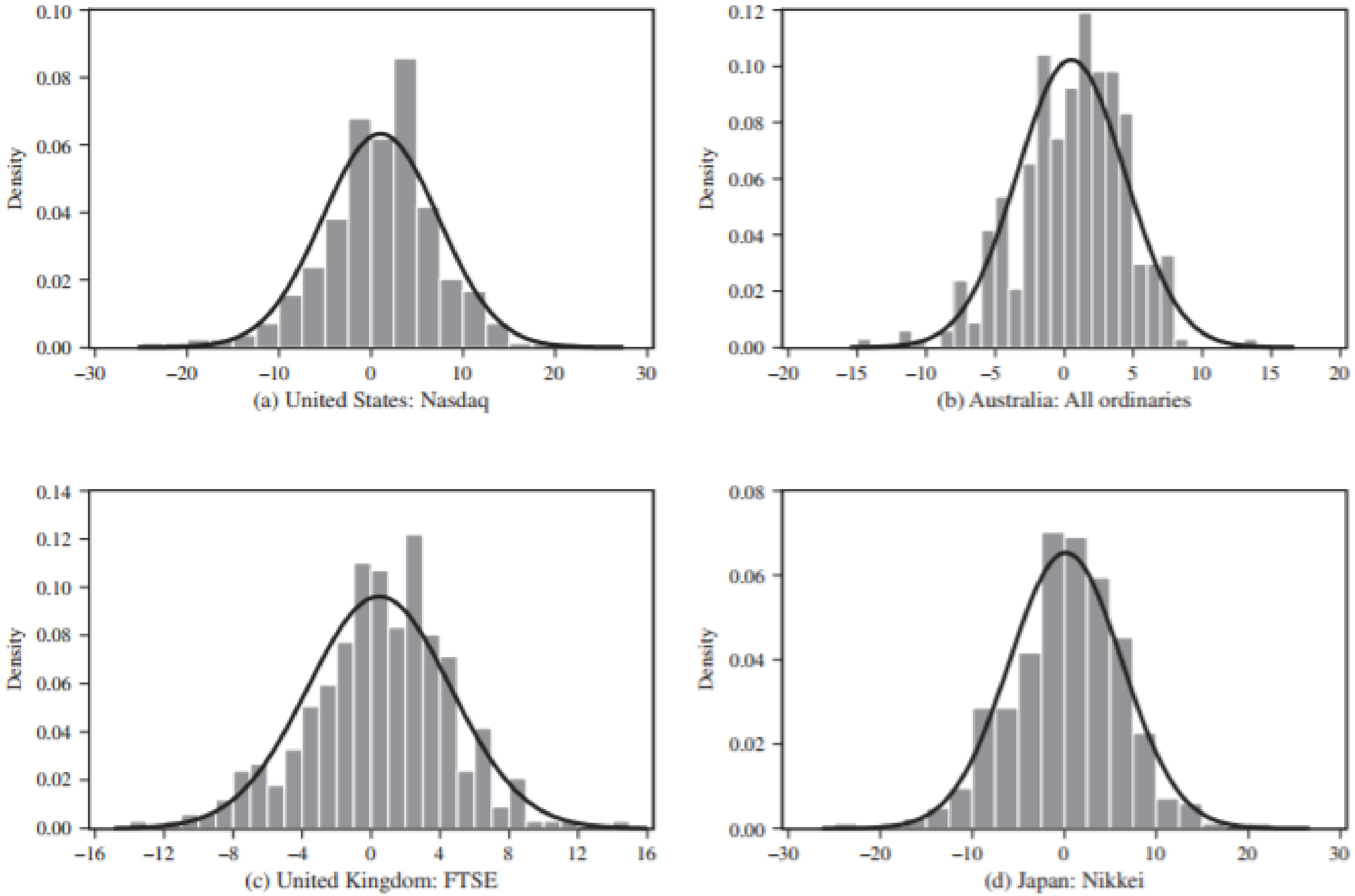


FIGURE 14.2 Histograms of returns to stock indices.



EXAMPLE 14.2 Simulating Time-Varying Volatility

- For Figure 14.3:
  - Note that relative to the series in the top panel, volatility in the bottom panel is not constant
  - It changes over time and it clusters—there are periods of small changes and periods of big changes
- For Figure 14.4
  - The second distribution has higher frequencies around the mean (zero) and higher frequencies in the tails (outside ± 3)

FIGURE 14.3 Simulated examples of constant and time-varying variances

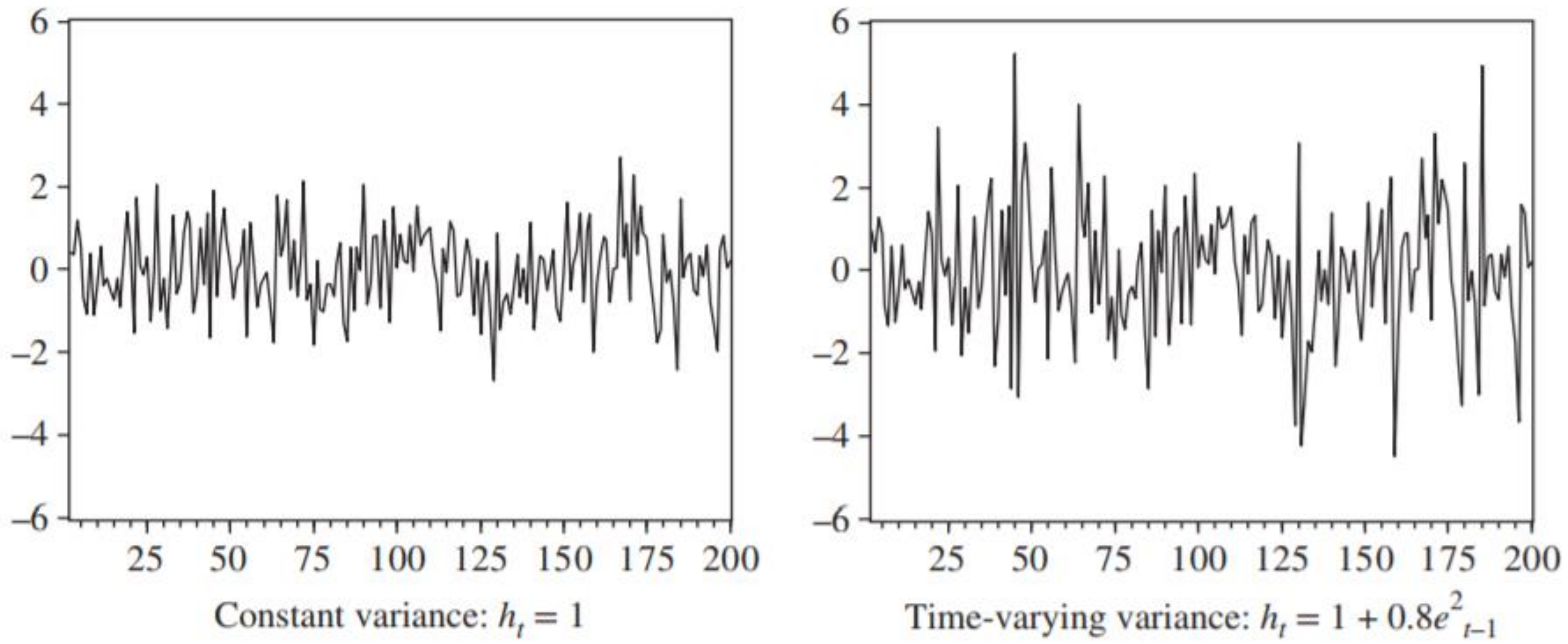
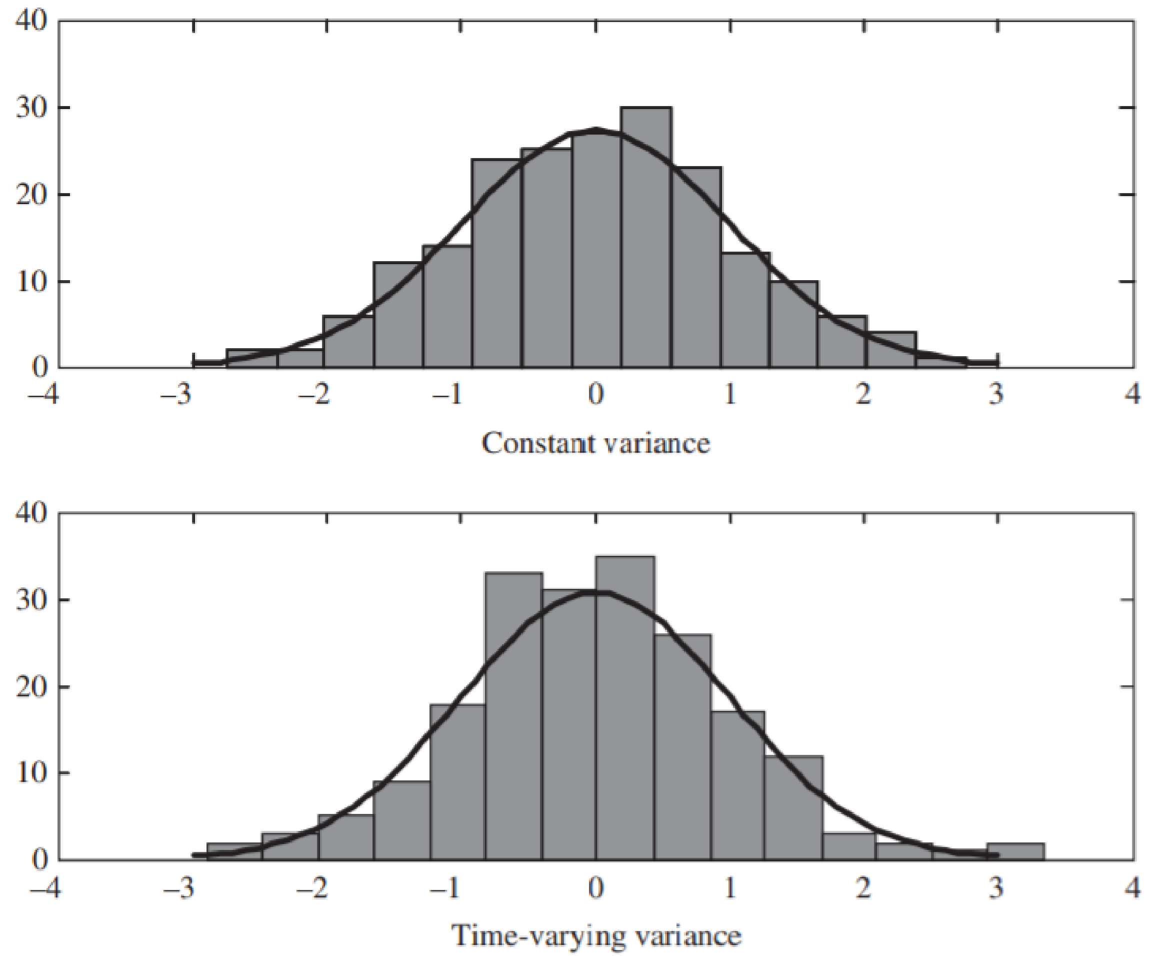


FIGURE 14.4 Frequency distributions of the simulated models



14.2 Time-Varying Volatility 2 of 2

- The ARCH model is intuitively appealing because it seems sensible to explain volatility as a function of the errors e_t
  - These errors are often called "shocks" or "news" by financial analysts
  - According to the ARCH model, the larger the shock, the greater the volatility in the series
  - This model captures volatility clustering, as big changes in e_t are fed into further big changes in h_t via the lagged effect e_{t-1}

14.3 Testing, Estimating, and Forecasting

- A Lagrange multiplier (LM) test is often used to test for the presence of ARCH effects

- To perform this test, first estimate the mean equation:
  - (14.3) $\hat{e}_t^2 = \gamma_0 + \gamma_1 \hat{e}_{t-1}^2 + v_t$

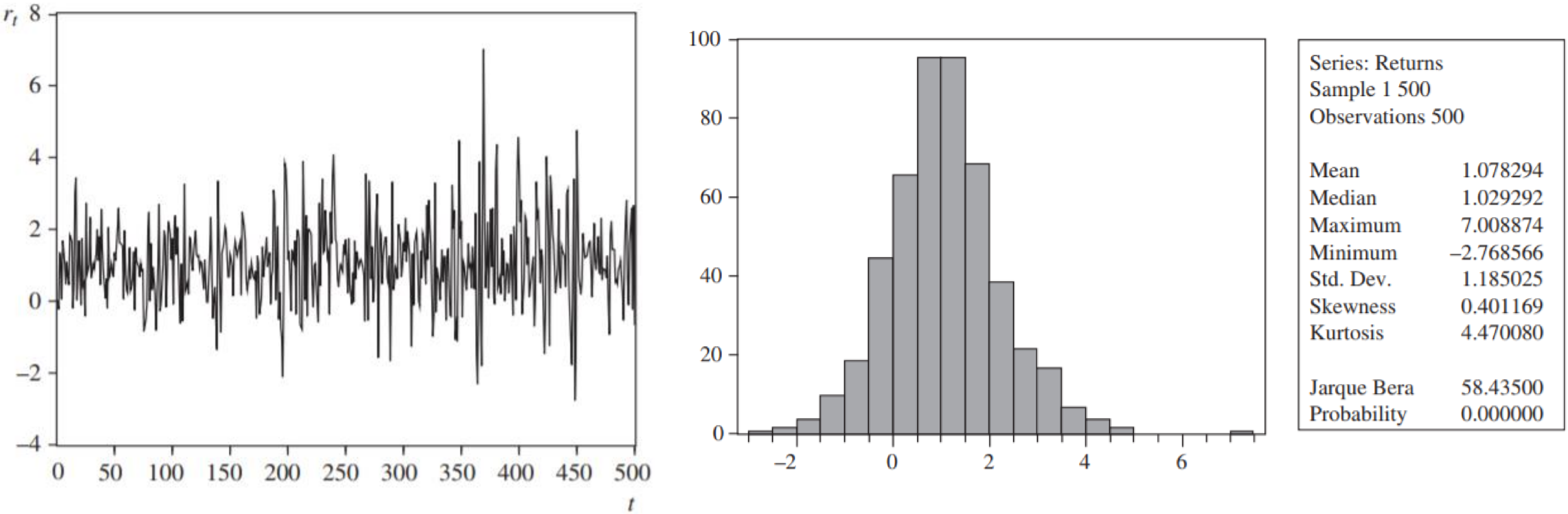
- The null and alternative hypotheses are:

$$H_0 : \gamma_1 = 0 \qquad H_1 : \gamma_1 \neq 0$$

EXAMPLE 14.3 Testing for ARCH in BYD Lighting 1 of 2

- Consider the returns from buying shares in the hypothetical company Brighten Your Day (BYD) Lighting
- The time series shows evidence of time-varying volatility and clustering, and the unconditional distribution is non-normal

FIGURE 14.5 Time series and histogram of returns for BYD lighting.



EXAMPLE 14.3 Testing for ARCH in BYD Lighting 2 of 2

- The results for an ARCH test are:

$$\hat{e}_t^2 = 0.908 + 0.353 e_{t-1}^2 \qquad R^2 = 0.124$$
$$(t) \qquad\qquad (8.409)$$

  - The t-statistic suggests a significant first-order coefficient
  - The sample size is 500, giving LM test value of $(T - q)R^2 = 61.876$
  - Comparing the computed test value to the 5% critical value of a $\chi^2_{(1)}$ distribution ($\chi^2_{(0.95, 1)} = 3.841$) leads to the rejection of the null hypothesis
    - The residuals show the presence of ARCH(1) effects

EXAMPLE 14.4 ARCH Model Estimates for (BYD) Lighting

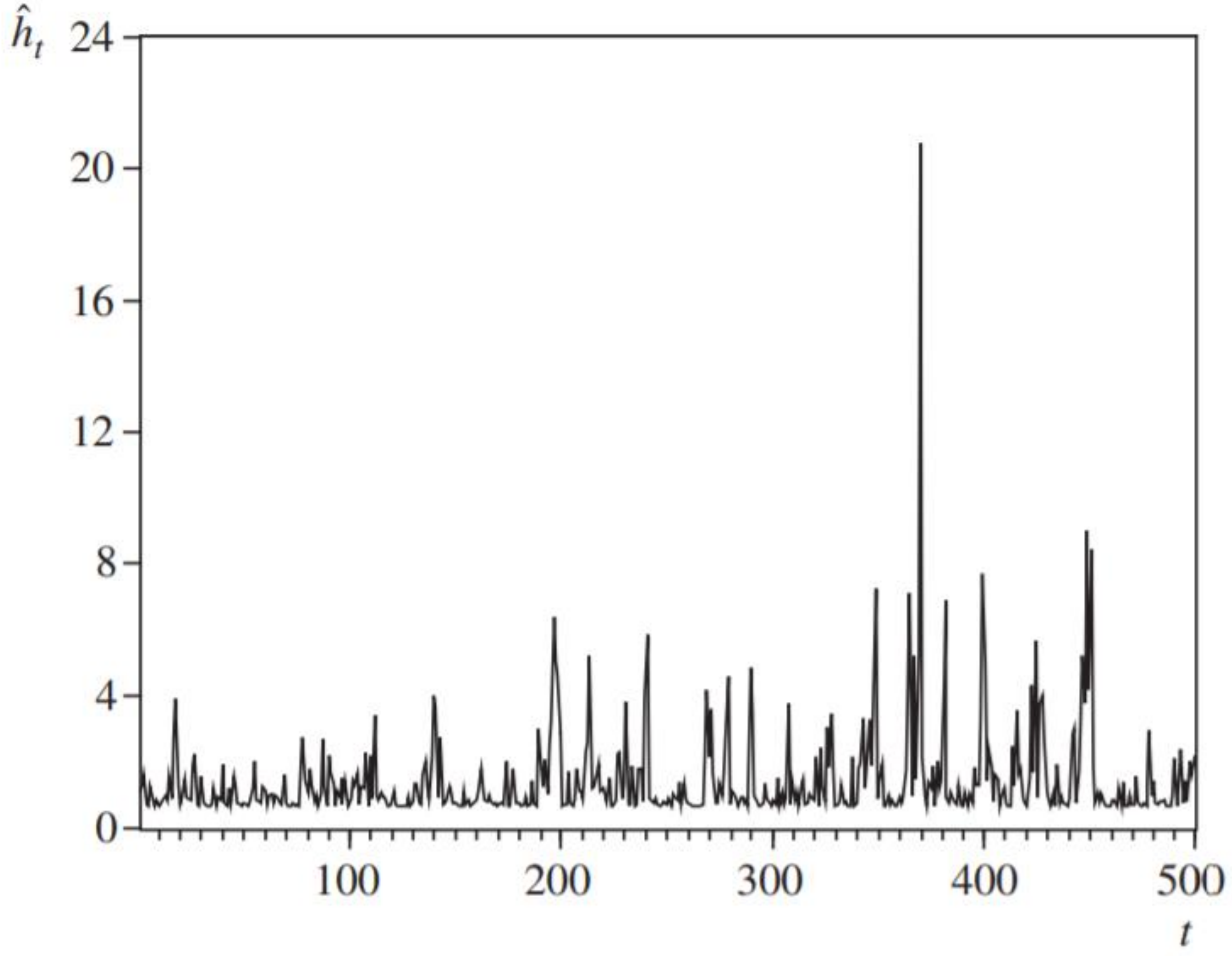
- ARCH models are estimated by the maximum likelihood method
- Equation (14.4) shows the results from estimating an ARCH(1) model applied to the monthly returns from buying shares in Brighten Your Day Lighting
- (14.4a) $\hat{r}_t = \hat{\beta}_0 = 1.063$
- (14.4b) $\hat{h}_t = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{e}_{t-1}^2 = 0.642 + 0.569 \hat{e}_{t-1}^2$

$$(t) \qquad\qquad\qquad\qquad (5.536)$$

EXAMPLE 14.5 Forecasting Brighten Your Day Volatility

- The forecast return and volatility are:
  - (14.5a) $\hat{r}_{t+1} = \hat{\beta}_0 = 1.063$
  - (14.5b) $\hat{h}_{t+1} = \hat{\alpha}_0 + \hat{\alpha}_1\left(r_t - \hat{\beta}_0\right)^2 = 0.642 + 0.569(r_t - 1.063)^2$
- Equation (14.5a) gives the estimated return that is both the conditional and unconditional mean return

FIGURE 14.6 Plot of conditional variance



14.4 Extensions

- The ARCH(1) model can be extended in a number of ways
  - One obvious extension is to allow for more lags
  - An ARCH(q) model would be:
    - (14.6) $h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 ... + \alpha_q e_{t-q}^2$
- Testing, estimating, and forecasting, are natural extensions of the case with one lag

14.4.1 The GARCH Model—Generalized ARCH 1 of 3

- One of the shortcomings of an ARCH(q) model is that there are $q + 1$ parameters to estimate
  - If q is a <u>large number</u>, we may lose accuracy in the estimation
  - The generalized ARCH model, or **GARCH**, is an alternative way to capture long lagged effects with <u>fewer parameters</u>

14.4.1 The GARCH Model—Generalized ARCH 2 of 3

- Consider Eq. 14.6 but write it as:
  - $h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \alpha_1 e_{t-2}^2 + \beta_1^2 \alpha_1 e_{t-3}^2 + \cdots$
  - Add and subtract $\beta_1 \alpha_0$and rearrange terms as follows:
  - $h_t = (\alpha_0 - \beta_1 \alpha_0) + \alpha_1 e_{t-1}^2 + \beta_1(\alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \cdots)$
  - Since, $h_{t-1} = \alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \beta_1^2 \alpha_1 e_{t-4}^2 + \cdots$
  - We may simplify to
  - (14.7) $h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1}$

14.4.1 The GARCH Model—Generalized ARCH 3 of 3

- This generalized ARCH model is denoted as GARCH(1,1)
  - The model is a very popular specification because it fits many data series well
  - It tells us that the volatility changes with lagged shocks (e^2_{t-1}) but there is also momentum in the system working via h_{t-1}
  - One reason why this model is so popular is that it can capture long lags in the shocks with only a few parameters

EXAMPLE 14.6 A GARCH Model for BrightenYourDay

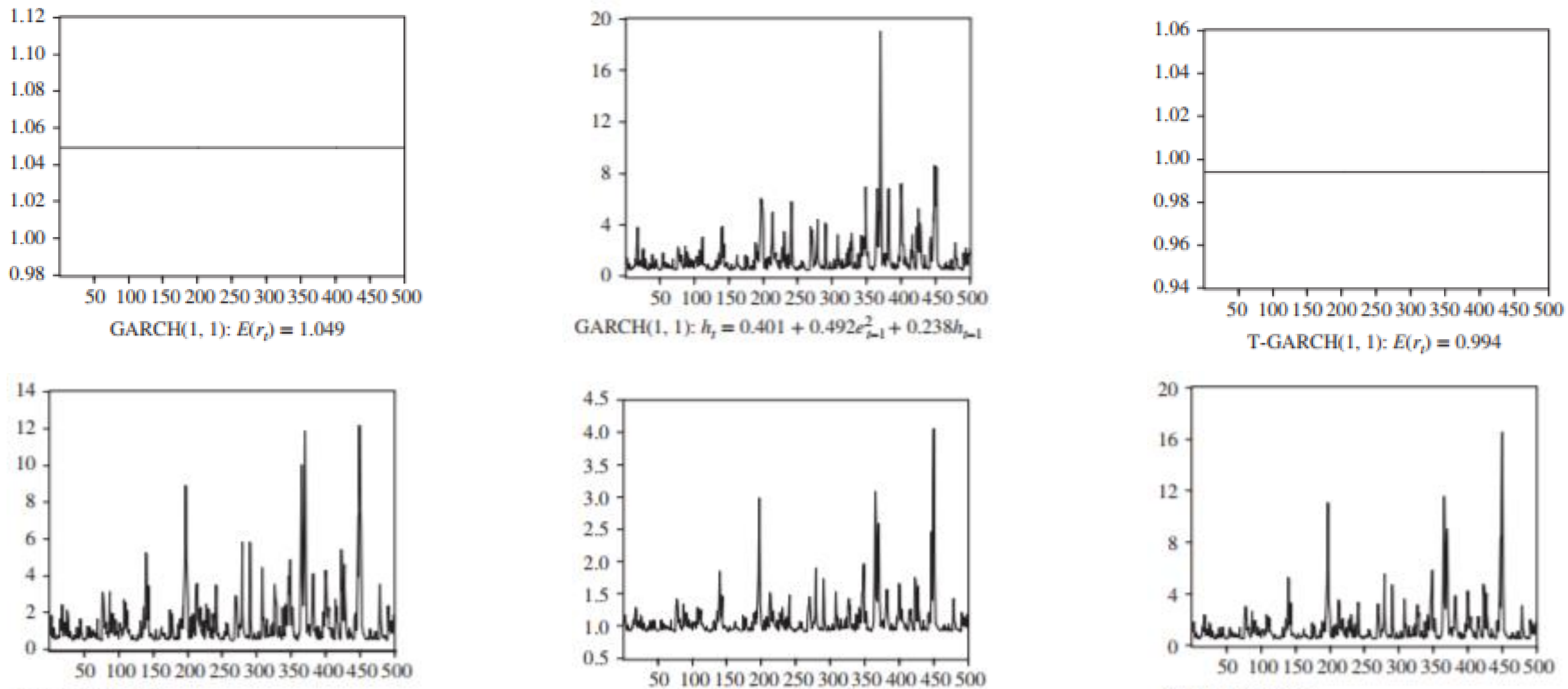
- Consider again the returns to our shares in Brighten Your Day Lighting, which we reestimate (by maximum likelihood) under the new model:

$$\hat{r}_t = 1.049$$

$$\hat{h}_t = 0.401 + 0.492\hat{e}_{t-1}^2 + 0.238h_{t-1}$$

$$(t) \qquad\qquad\qquad (4.834) \qquad\quad (2.136)$$

FIGURE 14.6 Plot of conditional variance



GARCH(1, 1): $E(r_t) = 1.049$

GARCH(1, 1): $h_t = 0.401 + 0.492e^2_{t-1} + 0.238h_{t-1}$

T-GARCH(1, 1): $E(r_t) = 0.994$

T-GARCH(1, 1): $h_t = 0.356 + (0.263 + 0.492d_{t-1})e^2_{t-1} + 0.287h_{t-1}$

GARCH-in-mean: $E(r_t) = 0.818 + 0.196h_t$

GARCH-in-mean: $h_t = 0.370 + (0.295 + 0.321d_{t-1})e^2_{t-1} + 0.278h_{t-1}$

14.4.2 Allowing for an Asymmetric Effect

- The threshold ARCH model, or **T-ARCH**, is one example where positive and negative news are treated asymmetrically
  - In the T-GARCH version of the model, the specification of the conditional variance is:

$$h_t = \delta + \alpha_1 e_{t-1}^2 + \gamma d_{t-1} e_{t-1}^2 + \beta_1 h_{t-1}$$

  - (14.8)

$$d_t = \begin{cases} 1 & e_t < 0 \text{ (bad news)} \\ 0 & e_t \geq 0 \text{ (good news)} \end{cases}$$

EXAMPLE 14.7 A **T-GARCH** Model for BYD

- The returns to our shares in Brighten Your Day Lighting were reestimated with a T-GARCH(1,1) specification:

$$\hat{r}_t = 0.994$$

$$\hat{h}_t = 0.356 + 0.263 e_{t-1}^2 + 0.492 d_{t-1} e_{t-1}^2 + 0.287 h_{t-1}$$
$$(t) \qquad (3.267) \qquad (2.405) \qquad (2.488)$$

- Overall, negative shocks create greater volatility in financial markets

14.4.3 GARCH-In-Mean and Time-Varying Risk Premium

- Another popular extension of the GARCH model is the "**GARCH-in-mean**" model
- (14.9a) $y_t = \beta_0 + \theta h_t + e_t$
- (14.9b) $e_t | I_{t-1} \sim N(0, h_t)$
- (14.9c) $h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1},$
  $\delta > 0,\ 0 \leq \alpha_1 < 1,\ 0 \leq \beta_1 < 1$

EXAMPLE 14.8 A GARCH-in-Mean Model for BYD 1 of 2

- The returns to shares in Brighten Your Day Lighting were reestimated as a GARCH-in-mean model:

- $$\hat{r}_t = 0.818 + 0.196 h_t$$
  $$(t) \qquad\qquad (2.915)$$

  $$\hat{h}_t = 0.370 + 0.295 e_{t-1}^2 + 0.321 d_{t-1} e_{t-1}^2 + 0.278 h_{t-1}$$
  $$(t) \qquad\qquad (3.426) \qquad (1.979) \qquad\qquad (2.678)$$

EXAMPLE 14.8 A GARCH-in-Mean Model for BYD 2 of 2

- The results show that as volatility increases, the returns correspondingly increase by a factor of 0.196
- In other words, this result supports the usual view in financial markets—high risk, high return

14.4.4 Other Developments 1 of 2

- The GARCH, TGARCH, and GARCH-in-mean models are three important extensions of the original ARCH concept
- The EGARCH model is: $\ln(h_t) = \delta + \beta_1 \ln(h_{t-1}) + \alpha \left| \frac{e_{t-1}}{\sqrt{h_{t-1}}} \right| + \gamma \left(\frac{e_{t-1}}{\sqrt{h_{t-1}}} \right)$
- Where $\left(\frac{e_{t-1}}{\sqrt{h_{t-1}}} \right)$ are the standardized residuals

14.4.4 Other Developments 2 of 2

- The leverage effect refers to the generally observed negative correlation between an asset return and its volatility changes
- Another significant development is to allow the conditional distribution of the error term to be non-normal
- Because empirical distributions of financial returns generally exhibit fat tails and clustering around zero, the t-distribution has become a popular alternative to the assumption of normality

Key Words

- ARCH
- ARCH-in-mean
- conditionally normal
- GARCH
- GARCH-in-mean
- T-ARCH and T-GARCH
- time-varying variance

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